Vector spaces

Three weeks

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Module

This chapter is the whole EV module: Vector spaces. It will take place both in the B3 and B4 bimesters.

The **prerequisites** are:

- Module [MATH-S1-1-OBM] from the seminar.
- Solving a linear system with two equations and two variables.
- Defining a vector plane or a vector line in the 3D space with Cartesian or parametric equations.

Context

Mathematical engineering involves modeling the situation we study. This situation is hence transposed into a mathematical description, which enables one to answer (some times only partially) to the questions of interest. For example, the evolution of an animal population can be modeled as a recursive sequence, the numbers of individuals at each generation depending on those at the previous generation. Similarly, the motion of an object falling with the gravity can be modeled as a function of the time fulfilling a specific differential equation.

It is important to keep in mind that, for a given situation, several models may be used. These models can formulate different points of view or different levels of complexity. The animal population could also be modeled with a differential equation or with a set of partial differential equations. The free fall motion's model could include the earth's rotation and the wind's speed, leading to a model which will be more complex than the one you have used in Newtonian physics.

From all the possible models, the simplest ones are the ***linear*** models. In these, the data can only be linked with linear dependence relations. You have handled such relations in your past, they were the simplest ones. Despite their simplicity, linear models allow solving a large number of standard computer-related problems. One can, for example, cite some of them:

1. reliable transmission of data through a communication channel, using linear error-correcting codes;
2. solving partial differential equations using numerical simulations;
3. getting a polynomial regression function in artificial intelligence: the purpose is to find the polynomial function whose graph will fit at best a set of data;
4. training neural networks with deep learning algorithms;
5. face recognition on photos, using principal components analysis.

The existing algorithms which enable one to solve these problems, as well as their underlying models, are skills that a computer scientist is expected to have.

In your mathematical history at EPITA, you have already seen two linear situations: about recursive sequences and differential equations.

A recursive sequence of order one $(u_n)_{n\in\mathbb{N}}$ is defined by an initial condition and by a relation $u_{n+1} = f(u_n)$ where f is a real function. You have seen that, for most of the functions f, it is not easy to get u_n as a function of n. Most of the time, it is just impossible. However, there is an easy case: when the function f is linear, that is, $f(x) = \alpha x$ for a value $\alpha \in \mathbb{R}$. In this case, u_{n+1} depends on u_n through a linear function and the sequence (u_n) is geometric. There exists a simple formula providing u_n as a function of n. This simplicity still remains if we work with higher order recursive sequences, provided that the involved relations are linear. Hence, in the Fibonacci sequence, u_{n+2} is a linear function of u_{n+1} and u_n. Similarly, an homogeneous linear differential equation consists in a linear relation between a function and its derivatives. Such an equation has explicit solutions, unlike many non-linear differential equations.

In order to be able to use the linear framework in professional contexts, we will have to study its language, the objects that underlie this framework and the interactions between these. Concepts that can be expressed using this language are at the heart of all the professional uses. The approach here is more abstract than what you have seen before. This abstraction comes from the great variability of contexts in which linear models are present. If we think in terms of object oriented programming, we can say that many situations inherit from the abstract class «vector space».

Learning outcomes

At the end of this chapter, you are expected to be able to:

- Recognize a vector space or a linear subspace.
- Identify the properties of a vector family (linearly independent?, spanning family?, basis?).
- Find the dimension of a linear subspace (when it exists).
- Use a dimension argument when reasoning with vector spaces.

More precisely, you should be able to:

- Prove whether a set is a vector space or not.
- Analyze the sum of two linear subspaces (direct sum?, supplementary?).
- Analyze a vector family in a basic vector space (linearly independent?, spanning family?, basis?).
- Find the coordinates of a vector in a given basis in finite dimension.
- Find a basis and the dimension of a linear subspace (in the case of finite dimension).
- Prove an inclusion or equality relation between two linear subspaces.
- Use a dimension argument to prove that two linear subspaces are equal (in the case of finite dimension).

1 Vectors and vector spaces

1.1 Summary

The purpose of this section is to start handling the mathematical objects called vectors. Not only those that you have seen until now, defined in the physical space by a length and a direction. But vectors in the enlarged sense, as you are currently discovering them. Those are just elements from a set with an additive and a multiplicative operations, fulfilling specific requirements, a set that is called a ***vector space***.

We will hence explore the following vector spaces, in which you will frequently work:

- $\mathbb{R}^n$, the set of n-tuples of real numbers. We will start with the familiar plane, when $n = 2$,
- the set of the numerical sequences,
- the set of the numerical functions from $\mathbb{R}$ to $\mathbb{R}$.

However, this list is not exhaustive, we will see other examples.

1.2 Exercises

Exercise 1.1
To start with in this module, let us work in the vector space $\mathbb{R}^2$. The purpose of the exercise is to *draw*. You can use the frame below (2 squares = 1 unit).

1. Draw the axes of the plane $\mathbb{R}^2$ and the vector $u = (3, 1)$.

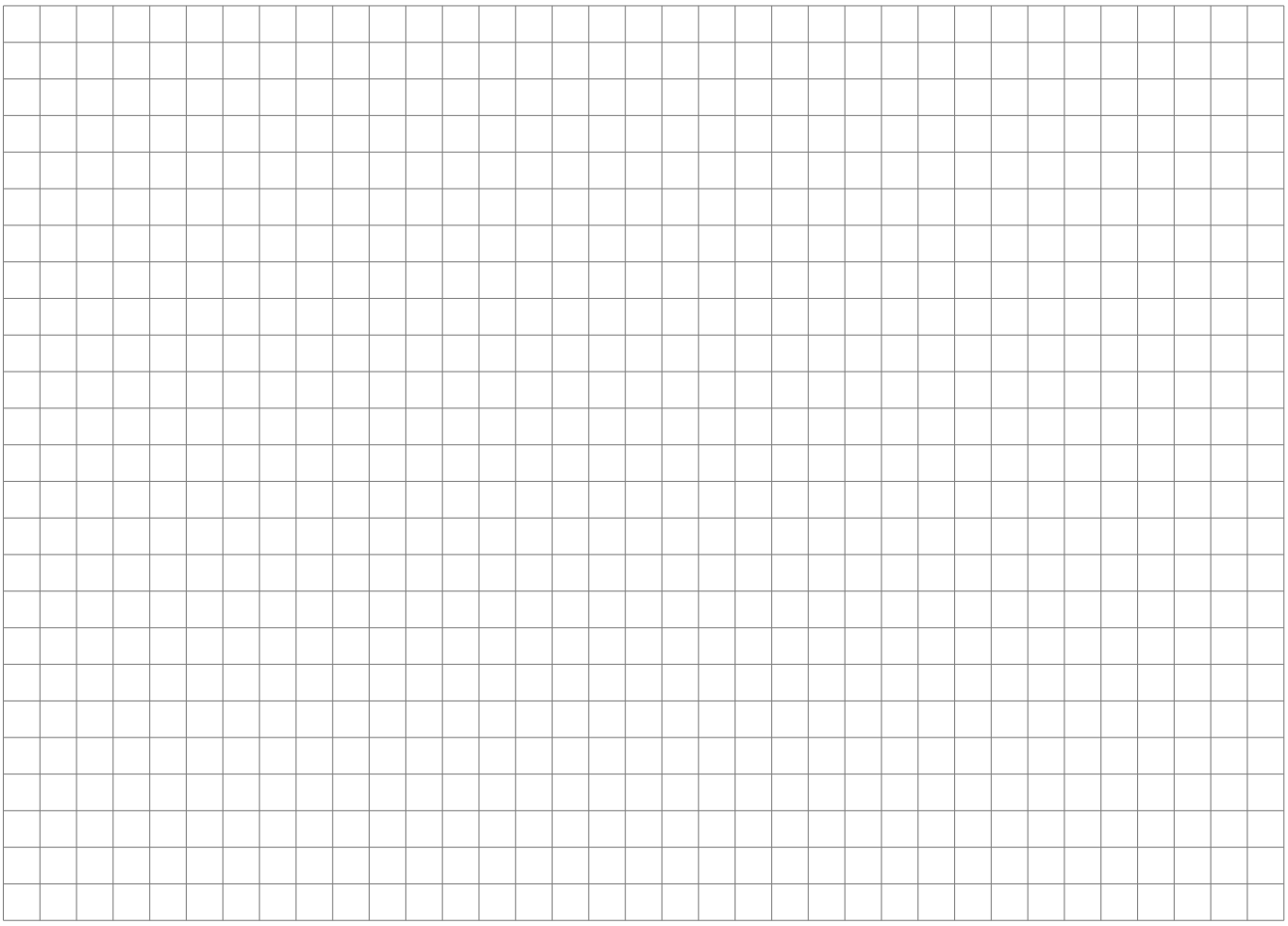
2. Draw the vectors $2u$, $-u$, $-\frac{1}{2}u$ and $0u$. Use the figure to find their coordinates.

   What is, in terms of geometry, the set $L = \{\alpha u, \alpha \in \mathbb{R}\}$?

3. On your figure, draw the vector $v = (-2, 1)$. Then find graphically the vectors $u + v$, $v - u$ and $2u + 3v$. Use the figure to find their coordinates.

   What is, in terms of geometry, the set $E = \{\alpha u + \beta v, (\alpha, \beta) \in \mathbb{R}^2\}$?

   Try to do similar graphical representations in $\mathbb{R}^3$. You may use the software *GeoGebra* in that purpose (it is easy to find on the web).



Exercise 1.2
(Homework) We focus on the set E of real arrays with 2 rows and 2 columns, of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{where } a, b, c \text{ and } d \text{ are real numbers}$$

Let us define on E the operations denoted by $+$ and $\cdot$ defined by

- for any A, B of the form $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$,

$$A + B = \begin{pmatrix} a + a' & b + b' \\ c + c' & d + d' \end{pmatrix}$$

- and for any $\alpha \in \mathbb{R}$

$$\alpha \cdot A = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

.

1. What would be the neutral element for the operations + and $\cdot$ that we have defined on E?
2. Check that the properties which define E to be a vector space over $\mathbb{R}$ are fulfilled.

Exercise 1.3

Let f and g be two real functions defined on an interval I from $\mathbb{R}$.

1. What is the function $f + g$?
2. Let E be the set of functions from I to $\mathbb{R}$. What is $\boldsymbol{0}_E$?
3. The addition of functions is sometimes said to be an internal operation over E. What is the sense of the word ***internal***? Define this operation as a mapping, specifying the input and output sets.
4. Let α be an arbitrary real number. What is the function αf ? The multiplication of functions by real numbers is sometimes said to be an external operation. Why ***external***? Define this operation as a mapping, specifying the input and output sets.
5. E is also denoted by $\mathbb{R}^I$. Do you think that E, together with the internal and external operations defined above, is a vector space over $\mathbb{R}$?

Exercise 1.4

We focus on the set $\mathbb{R}^{\mathbb{N}}$ of the real sequences. Let us define on $\mathbb{R}^{\mathbb{N}}$ an external operation as follows: for any $\alpha \in \mathbb{R}$ and any $(u_n)_{n\in\mathbb{N}}$,

$$\alpha \cdot (u_n) = (\alpha^n u_n) = (u_0, \alpha u_1, \alpha^2 u_2, \alpha^3 u_3, \ldots).$$

1. What would be the neutral element for this operation?
2. If we also use the usual addition operation on $\mathbb{R}^{\mathbb{N}}$, do we get a vector space over $\mathbb{R}$?

Exercise 1.5

Enumerate all the reference vector spaces that you have seen until now. For each of them, specify the + and . operators and the zero-vector.

2 Linear subspaces

2.1 Summary

You have studied the eight properties which, when satisfied by a set E and by its internal and external operations (addition and scalar multiplication), make a vector space. However, we will most of the time work in a restricted collection of vector spaces: in $\mathbb{R}^n$, in the set of functions from $\mathbb{R}$ to $\mathbb{R}$, in $\mathbb{R}[X]$ or in $\mathbb{R}^{\mathbb{N}}$.

This list is not exhaustive, but the most common way to define other vector spaces is the definition of a linear subspace of one of these basic vector spaces. A linear subspace is a subset which is also a vector space.

We will see later how we can ***build*** such a linear subspace and how we can avoid redundancy in this definition.

2.2 Exercises

Exercise 2.6
The purpose of this exercise is to study the linear subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$.

1. To start with, let us focus on $\mathbb{R}^2$. It will be easy and recommended to ***draw***.

   The vector $(0,0)$ is denoted by O.

   (a) Represent the subsets of $\mathbb{R}^2$ defined below and, for each of them, say whether it is a linear subspace. If it is not, explain why.

   A : Singleton $\{O\}$
   B : Disk of radius 1 and center O
   C : Half-plane $x \geq 0$
   D : (Ox) axis
   E : Line $x + y = 0$
   F : Line $x + y = 1$

   (b) Is it possible that two linear subspaces of the same vector space are disjoint?

   (c) What are the linear subspaces of $\mathbb{R}^2$?

2. What are the linear subspaces of $\mathbb{R}^3$.

Exercise 2.7

1. What is the set $E = \{aX^2 + bX + c, (a,b,c) \in \mathbb{R}^3\}$?
2. Using a similar expression, define the set F of polynomials of degree exactly equal to two.
3. Which inclusion relation between E and F? In which reference vector space are they included?
4. What is the null polynomial? Is it an element of E? Of F?
5. Are E and F closed for the multiplication . ? For the addition +?

Exercise 2.8
Are the following sets $\mathbb{R}$-vs?

1. $F = \{(x,y,z) \in \mathbb{R}^3,\ x + y + z = 0\}$
2. $F = \{(x,y,z) \in \mathbb{R}^3,\ x + y + z = 1\}$
3. ***(Homework)*** $F = \{(x,y,z) \in \mathbb{R}^3,\ xy = 0\}$
4. $F = \{(x,y,z) \in \mathbb{R}^3,\ x^2 = z^2\}$
5. $F = \{P \in \mathbb{R}[X], XP'' - 3P = 0\}$
6. $F = \{P \in \mathbb{R}[X],\ P'(3) = 0\}$
7. ***(Homework)*** $F = \{(u_n) \in \mathbb{R}^{\mathbb{N}},\ (u_n) \text{ convergent}\}$
8. $F = \{(u_n) \in \mathbb{R}^{\mathbb{N}}, \forall n \in \mathbb{N}, u_{n+1} = 3u_n - 1\}$
9. $F = \{(u_n) \in \mathbb{R}^{\mathbb{N}},\ (u_n) \text{ divergent}\}$
10. $F = \{(u_n) \in \mathbb{R}^{\mathbb{N}}, \forall\, n \in \mathbb{N},\ u_{n+2} = 2u_{n+1} - u_n\}$
11. $F = \{(u_n) \in \mathbb{R}^{\mathbb{N}}, \forall\, n \in \mathbb{N},\ u_{n+2} = 2u_{n+1} - u_n^2\}$

12. ***(Homework)*** $F = \{(u_n) \in \mathbb{R}^{\mathbb{N}}, \exists\,(\alpha,\beta,\gamma) \in \mathbb{R}^3, \forall\, n \in \mathbb{N},\ u_n = \alpha 2^n + \beta 3^n + \gamma 4^n\}$
13. $F = \{f \in \mathbb{R}^{\mathbb{R}},\ f \text{ is an increasing function}\}$
14. $F = \{f \in \mathbb{R}^{\mathbb{R}},\ f(0) = 0\}$
15. ***(Homework)*** $F = \{f \in \mathbb{R}^{\mathbb{R}},\ f \text{ is an even function}\}$
16. $F = \{f \in \mathbb{R}^{\mathbb{R}},\ \lim\limits_{+\infty} f = +\infty\}$
17. ***(Homework)*** $F = \{f \in \mathbb{R}^{\mathbb{R}},\ f \text{ is continuous}\}$
18. $F = \{f \in \mathcal{C}^0([a,b],\mathbb{R}),\ \int_a^b f(t)\,\mathrm{d}t = 0\}$

Exercise 2.9
Let F and G be two linear subspaces of a $\mathbb{R}$-vs E. Study the property

$$F \cup G \text{ linear subspace of } E \iff F \subset G \text{ or } G \subset F$$

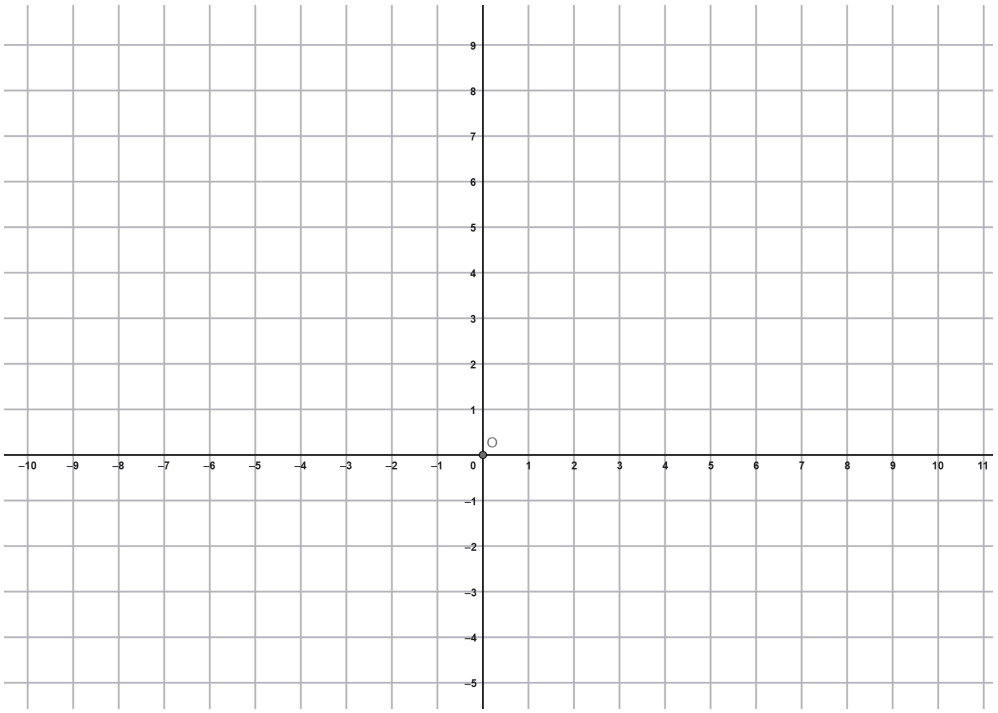
Exercise 2.10
In the vector space $E = \mathbb{R}^2$, on consider the vectors: $u = (2,4)$, $v = (3,1)$ and $w = (-1,-2)$.

1. Consider the set $F = \{(x,y) \in \mathbb{R}^2, \exists\,\alpha \in \mathbb{R}, (x,y) = \alpha u\}$. Show that F is a linear subspace of $\mathbb{R}^2$.

2. Consider the following sets:

$$G = \Big\{(x,y) \in \mathbb{R}^2, \exists\,\alpha \in \mathbb{R}, (x,y) = \alpha v\Big\} \text{ and } H = \Big\{(x,y) \in \mathbb{R}^2, \exists\,\alpha \in \mathbb{R}, (x,y) = \alpha w\Big\}$$

   (a) Are G and H some $\mathbb{R}$-vector spaces?
   (b) What are these 3 sets in terms of geometry? Draw them.



   (c) Let $X = (-2,6)$. Graphically, say whether $X \in F + G$ and whether $X \in F + H$.
   (d) What can be said about $F + G$? And about $F + H$?

N.B.: as a summary and to correct an ambiguity in the MiMos, here are the definitions of "direct sum" and "supplementary" (you have to know them):

Definition : let E be a $\mathbb{R}$-vector space, F and G two linear subspaces of E.

- F and G are in **direct sum** if $F \cap G = \{0_E\}$.
- F and G are **supplementary in** E if, on the one hand, $E = F + G$ and on the other hand, F and G are in direct sum. In this situation, we write $E = F \oplus G$.

  In other words, $E = F \oplus G \iff (E = F + G \text{ and } F \cap G = \{0_E\})$

Exercise 2.11
Let E be a $\mathbb{R}$-vs, let F, G and H be three linear subspaces of E such that $F \oplus G = E$ and $F \oplus H = E$. Can we deduce that $G = H$? Justify your answer, using the previous exercise.

Exercise 2.12
In $E = \mathbb{R}^3$, consider the three linear subspaces:

$$F = \left\{(x, y, z) \in \mathbb{R}^3,\, x + y = 0\right\}, \quad G = \left\{(x, y, z) \in \mathbb{R}^3,\, z = 0\right\} \quad \text{and} \quad H = \left\{(x, y, z) \in \mathbb{R}^3,\, y = z = 0\right\}$$

1. True or false: $F \cap G = \{0_E\}$? $F \cap H = \{0_E\}$?
2. True or false: $F \oplus G = \mathbb{R}^3$? $F \oplus H = \mathbb{R}^3$? Justify.

★ **Exercice 2.13**
We work in the vector space $\mathbb{R}^{\mathbb{R}}$, the set of functions from $\mathbb{R}$ to $\mathbb{R}$.

1. What is the definition of an even and an odd function?
2. Let $\mathcal{E}$ et $\mathcal{O}$ denote the sets of even and odd functions. They are linear subspaces of $\mathbb{R}^{\mathbb{R}}$. What can you say about their sum?
3. Let A be a subset of $\mathbb{R}$ and $E_A \subset \mathbb{R}^{\mathbb{R}}$ the set of functions which vanish on A. Find a supplementary subspace of E_A in $\mathbb{R}^{\mathbb{R}}$.
4. How can we adapt the previous questions in $\mathbb{R}[X]$ and in $\mathbb{R}_3[X]$?

3 Spanning families, independent families and bases

3.1 Summary

Many questions encountered in linear algebra imply the definition of a vector space as a small subspace of a greater one. For example, the solution set of an homogeneous differential equation is a subspace of the space of all the differentiable functions. It is hence important to find a concise way to define such a subspace, that is, a way which requires the less external information. This is the purpose of a ***basis***.
This notion is specially useful in the framework of computer science, as it enables one to replace the whole subspace (which is infinite) with a ***finite*** family, which can hence be stored in a machine.

In this section, we learn how to find minimal families of vectors, with no redundancy, which span the whole space. These families are bases. We also learn how to detect smaller families, too small to span the whole space, but with no redundancy (independent families), or larger families, large enough to span the whole space but with redundancies (the spanning families). These notions enable one to define the number of «degrees of freedom» in a set of vectors.

3.2 Exercises

Exercise 3.14

For each of the following families, say whether it is linearly independent or not.

1. $\mathcal{F}_1 = \big(u = (1,0), v = (0,2), w = (-3,4)\big)$ in $\mathbb{R}^2$.
2. $\mathcal{F}_2 = (v_1, v_2, v_3)$ in $\mathbb{R}^3$, where $v_1 = (1,-1,1)$, $v_2 = (0,1,2)$ and $v_3 = (1,0,3)$.
3. $\mathcal{F}_3 = (w_1, w_2)$ in $\mathbb{R}^3$, where $w_1 = (2,-1,3)$ and $w_2 = (-8,4,12)$.
4. $\mathcal{F}_4 = (u_1, u_2, u_3)$ in $\mathbb{R}^3$, where $u_1 = (5,2,-11), u_2 = (1,1,2), u_3 = (-1,0,3)$.
5. ***(Homework)*** $\mathcal{F}_5 = (u_1, u_2, u_3)$ in $\mathbb{R}^3$, where $u_1 = (1,1,1), u_2 = (1,-1,0), u_3 = (-1,1,-1)$.
6. $\mathcal{F}_6 = (P, Q, R)$ in $\mathbb{R}[X]$ where: $P(X) = 1-X+X^2, Q(X) = 3+X-2X^2, R(X) = -1-3X+4X^2$
7. ***(Homework)*** $\mathcal{F}_7 = (P, Q, R)$ in $\mathbb{R}[X]$ where: $P(X) = 1, Q(X) = X^2+X+1, R(X) = X^2-X+1$
8. ($\star$) $\mathcal{F}_8 = \big((u_n), (v_n), (w_n)\big)$ in $\mathbb{R}^{\mathbb{N}}$ where: $(u_n) = (2^n), (v_n) = (3^n)$ and $(w_n) = (4^n)$.
9. ($\star$) $\mathcal{F}_9 = (f, g, h)$ in $\mathbb{R}^{\mathbb{R}}$, defined by: $\forall x \in \mathbb{R}$, $f(x) = e^x$, $g(x) = xe^x$ and $h(x) = e^{-x}$.

Exercise 3.15

1. Let E be a $\mathbb{R}$-vector space and $\mathcal{F} = (e_1, e_2, \ldots, e_n)$ a family of n non-zero vectors.

   Express mathematically the set C of the linear combinations of $\mathcal{F}$ vectors.

   Do you think that C is a linear subspace of E? How is C denoted?

2. In $\mathbb{R}^3$, consider the vectors $u = (1,1,0)$, $v = (-1,2,1)$ and $w = (-2,-2,0)$, and the sets:

   $$F = \{\alpha u,\ \alpha \in \mathbb{R}\},\ G = \Big\{\alpha u + \beta v,\ (\alpha,\beta) \in \mathbb{R}^2\Big\} \text{ and } H = \Big\{\alpha u + \beta w,\ (\alpha,\beta) \in \mathbb{R}^2\Big\}$$

   (a) Are F, G and H vector spaces over $\mathbb{R}$?
   (b) Express F, G and H using the Span notation. What are these sets, in terms of geometry?
   (c) Compare (regarding the inclusion relation) F and G. Then compare F and H.

3. In $\mathbb{R}^2$, draw $\mathrm{Span}\big((1,1)\big)$, $\mathrm{Span}\big((-1,1)\big)$, $\mathrm{Span}\big((1,2),(3,4)\big)$, $\mathrm{Span}\big((1,1),(2,2)\big)$ and $\mathrm{Span}\big((1,1),(-1,1),(3,4)\big)$.

4. In $\mathbb{R}^3$, what are (in term of geometry) the subspaces:
   $\mathrm{Span}\big((1,1,1)\big)$? $\mathrm{Span}\big((1,2,0),(0,3,4)\big)$? $\mathrm{Span}\big((1,1,0),(2,2,0)\big)$?

5. In $\mathbb{R}_3[X]$, let $F = \mathrm{Span}(1, X + X^2)$.
   True or false: $3 \in F$? $4 + 3X + 4X^2 \in F$? $4 + 3X + 3X^2 \in F$? Express F in intention.

6. ($\star$) In $\mathbb{R}^{\mathbb{R}}$, consider the functions $f : x \longmapsto 1$ and $g : x \longmapsto x$. How do you call $F = \mathrm{Span}(f, g)$?

Exercise 3.16
In $\mathbb{R}^3$, consider the family $\mathcal{F} = \big(u_1{=}(1,-1,2), u_2{=}(-1,4,1), u_3{=}(0,3,3)\big)$.

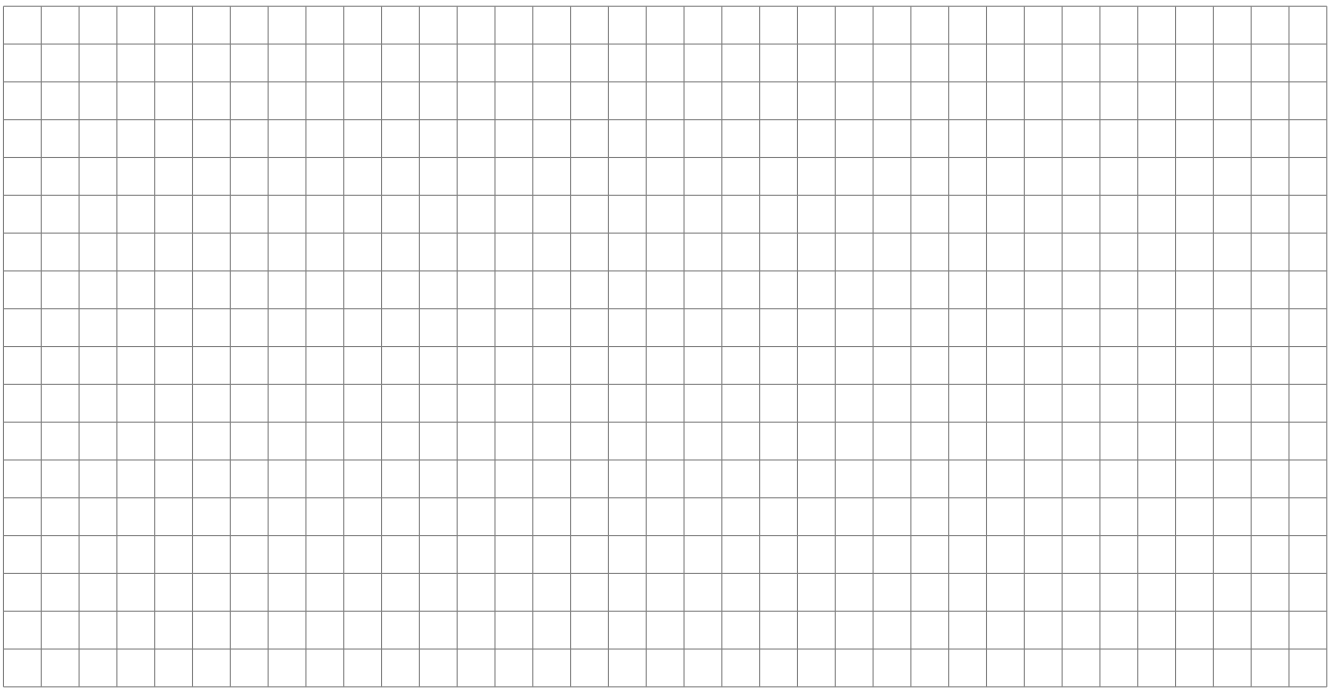
1. Show that $\mathcal{F}$ is linearly dependent. Extract a maximal independent subfamily (the latter will be denoted by $\mathcal{F}'$).
2. Show that $\mathrm{Span}\mathcal{F} = \mathrm{Span}\mathcal{F}'$.
3. What is $\dim\big(\mathrm{Span}\mathcal{F}'\big)$?

Exercise 3.17
Find a spanning family of the following linear subspaces, then deduce **rigorously** their dimensions.

1. $E = \big\{(x,y) \in \mathbb{R}^2 \text{ such that } x - 2y = 0\big\}$
2. $F = \big\{(x,y,z) \in \mathbb{R}^3 \text{ such that } x - y = 0 \text{ and } z = 0\big\}$
3. $G = \big\{(x,y,z) \in \mathbb{R}^3 \text{ such that } x - y = 0\big\}$
4. $H = \big\{(x,y,z) \in \mathbb{R}^3 \text{ such that } x - y + z = 0\big\}$
5. $I = \big\{(x,y,z,t) \in \mathbb{R}^4 \text{ such that } x - z = 0 \text{ and } 3x + 2y + 3z = 0\big\}$
6. ***(Homework)***
   $J = \big\{(x,y,z,t) \in \mathbb{R}^4 \text{ such that } x + y + t = 0 \text{ and } x + y + z = 0\big\}$
7. $K = \{P \in \mathbb{R}_2[X],\ P(0) = 0\}$
8. $L = \{P \in \mathbb{R}_2[X],\ P(0) = 0 \text{ and } P'(1) = 0\}$
9. ***(Homework)*** $M = \{P \in \mathbb{R}_3[X],\ X^2 \,|\, P\}$

Exercise 3.18
1. What are the standard bases of $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}_n[X]$? Deduce the dimensions of these vector spaces.
2. In $\mathbb{R}^2$, what can you say about the cardinal of a spanning family? Of an independent family? Of a dependent family? Can these properties be extended to vector spaces of dimensions $n > 2$?

Exercise 3.19
1. Show that $\mathcal{B} = \big(\varepsilon_1{=}(2,1), \varepsilon_2{=}(-1,1)\big)$ is a basis of $\mathbb{R}^2$.
2. Let $u = (5,1)$. Find graphically the coordinates of u in basis $\mathcal{B}$. **Your figure must show how you get them.**



3. Check your answer with a computation.

Exercise 3.20

1. Let $E = \mathbb{R}^3$. Is the family $\mathcal{F} = \big(u_1{=}(1,1,-1), u_2{=}(-1,1,1), u_3{=}(1,-1,1)\big)$ a basis of E? If it is, find the coordinates in this basis of vector $u = (1,2,3)$.

2. Let $E = \mathbb{R}_2[X]$. Show that $\mathcal{F} = \big(P_1 = 1, P_2 = X+1, P_3 = (X+1)^2\big)$ is a basis of E. Find the coordinates in this basis of $P = X + 2$.

Exercise 3.21

Let E be a $\mathbb{R}$-vector space of dimension 4. Let $\mathcal{F}$ be a family of E.

In the following situations, circle **all** all the properties which are possible ("linearly independent", "spanning family" and/or "linearly dependent").

a. If $\text{Card}(\mathcal{F}) = 3$, $\mathcal{F}$ can be: linearly independent spanning family linearly dependent

b. If $\text{Card}(\mathcal{F}) = 5$, $\mathcal{F}$ can be: linearly independent spanning family linearly dependent

c. If $\text{Card}(\mathcal{F}) = 6$, $\mathcal{F}$ can be: linearly independent spanning family linearly dependent

d. If $\text{Card}(\mathcal{F}) = 4$, $\mathcal{F}$ can be: linearly independent spanning family linearly dependent

Exercise 3.22

In the examples below, F and G are two linear subspaces of $E = \mathbb{R}^3$. Are they supplementary in E?

1. $E = \mathbb{R}^3$, $F = \{(x,y,z) \in \mathbb{R}^3, x - y + z = 0\}$, $G = \{(x,y,z) \in \mathbb{R}^3, x = y = z\}$

2. $E = \mathbb{R}^3$, $F = \{(x,y,z) \in \mathbb{R}^3, x - y + z = 0\}$, $G = \{(x,y,z) \in \mathbb{R}^3, z = 0\}$

3. $E = \mathbb{R}^3$, $F = \{(x,y,z) \in \mathbb{R}^3, x - y = z = 0\}$, $G = \{(x,y,z) \in \mathbb{R}^3, x - z = y = 0\}$

★ **Exercice 3.23**

In $\mathbb{R}^3$, consider the linear subspaces $F = \{(x, y, z) \in \mathbb{R}^3, y = x + z\}$ and $G = \{(x, y, z) \in \mathbb{R}^3, z = 0\}$.

1. Show that $F + G = \mathbb{R}^3$, but that F and G are not supplementary.
2. Find F' such that $F' \subset F$ and $F' \oplus G = \mathbb{R}^3$.